

New perspectives on Hawking radiation

Matteo Smerlak^{1,*} and Suprit Singh^{2,†}

¹*Max-Planck-Institut für Gravitationsphysik, Am Mühlenberg 1, D-14476 Golm, Germany*

²*Inter-University Centre for Astronomy and Astrophysics, Ganeshkhind, Pune 411 007, India*

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We develop an adiabatic formalism to study the Hawking phenomenon from the perspective of Unruh-DeWitt detectors moving along non-stationary, non-asymptotic trajectories. When applied to *geodesic* trajectories, this formalism yields surprising results: (i) though they have zero acceleration, the temperature measured by detectors on circular orbits is *higher* than that measured by static detectors at the same distance from the hole, and diverges on the photon sphere, (ii) in the near-horizon region, both outgoing *and* incoming modes excite infalling detectors, and, for highly bound trajectories ($E \ll 1$), the latter actually *dominate* the former, (iii) in this region, the relationship between the temperature of Hawking radiation and the relative velocity between the detector and the hole is *not* of Doppler type. We confirm the apparent perception of high-temperature *ingoing* Hawking radiation by infalling observers with $E \ll 1$ by a flux computation. We close by a discussion of the role played by spacetime curvature on the near-horizon Hawking radiation.

I. INTRODUCTION

Hawking has famously predicted [1, 2] that the gravitational collapse of a (say spherically symmetric) matter distribution of mass M will be perceived by observers at future infinity in the form of a stationary, outgoing, thermal flux of massless particles with temperature $T_H = (8\pi M)^{-1}$. Like most general-relativistic effects, this “Hawking phenomenon” is *a priori* observer-dependent, and it is not immediately clear what other, non-asymptotic observers would measure. To investigate this question, Unruh introduced in [3] a most useful and practical approach, based on a simple particle detector model (referred to as an “Unruh-Dewitt (UDW) detector”). In the case of static UDW detectors at “radius” r in a Schwarzschild spacetime, for instance, one finds [4] that the temperature measured by UDW detectors is $T_H(1 - 2M/r)^{-1/2}$; this shows that Hawking radiation satisfies the Tolman equilibrium condition [5], just like a normal thermal bath.

Another interesting class of trajectories worth investigating are *infalling geodesics*. Since these are not tangent to a timelike Killing field, however, one cannot use any Tolman-like *a priori* argument to infer the temperature measured by UDW detectors along these trajectories, and in particular when they cross the Schwarzschild horizon. Would they record Hawking radiation there? Or would they not, because Hawking radiation is created at some distance away from the black hole [6]?

Unruh gave an answer to this question in [3]. Taking his cues from the geometric similarity between the Schwarzschild and Rindler horizons, and the observation that the Unruh temperature $T_U = a/2\pi$ of geodesic observers in Rindler spacetime is zero (because their acceleration a is zero), he argued that “a geodesic detector near the horizon will not see the Hawking flux of particles”. This conclusion—although not supported by an explicit calculation in [3]—fits with the general view put forward in that paper that, near the horizon, the in-vacuum of gravitational collapse is not different from the Minkowski vacuum. The reduction of the (curved spacetime) Hawking effect to the (flat spacetime) Unruh effect in the near-horizon region [7] is further supported by the following fact: the Hawking temperature $T_H(1 - 2M/r)^{-1/2}$ perceived by a static observer at a radius $r > 2M$ in Schwarzschild spacetime approaches the Unruh temperature $(M/r^2)(1 - 2M/r)^{-1/2}/2\pi$ for an observer with the same acceleration in Minkowski spacetime when $R \rightarrow 2M$. Singleton and Wilburn have described this fact by saying that “the equivalence principle is restored at the horizon” [4].

For all that, heuristic arguments as well as actual computations have recently challenged this conclusion. In the former category, Helfer has argued [8] that “the vicinity of a black hole is a region in which essentially quantum-gravitational, Planck-scale, physics must dominate”; for Almheiri *et al.* [9], the principle of information conservation and other quantum-information-theoretic constraints suggest the presence of a “firewall” at the horizon. While interesting in themselves, these arguments are unfortunately too sketchy to lead to a definite prediction concerning the fate of horizon-crossing geodesic

* smerlak@aei.mpg.de

† suprit@iucaa.ernet.in

detectors. Following a more conventional semi-classical approach, Barbado *et al.* [10] have recently devised a framework, based on the notion of “effective temperature”, to study the nature of Hawking radiation near the horizon more explicitly. (The “effective temperature” introduced there is meant to give an estimation of the response of UDW detectors at any given point along their trajectory.) Somewhat surprisingly, they find that the effective temperature of detector dropped with zero initial velocity from infinity will rise, until it reaches the value $4T_H$ when crossing the horizon. This result appears to contradict Unruh’s conclusion [3], and does not accredit the notion that the Hawking effect reduces to the Unruh effect in the near-horizon region.

Very appealing for its conceptual simplicity, the “effective temperature” approach of [10] has several technical limitations which could legitimate some skepticism about these conclusions. First, the actual response function of a trans-horizon geodesic detector is *not* explicitly evaluated in [10], nor is any solid argument given to the effect that the “effective temperature” defined by these authors is indeed the one measured by an Unruh-DeWitt detector. Second, the framework used in [10], being entirely based on the retarded Eddington-Finkelstein time coordinate u , does not apply to the interior region of the black hole; hence one could speculate that the effect found in [10] is perhaps an artifact of the horizon being singular with respect to the u -coordinate. Third, the vacuum state used in the definition of the “effective temperature” function in [10] is not the standard Unruh vacuum, but a new, unconventional state in the extended Schwarzschild geometry. One may fear that this new state is perhaps unsuitable to describe the physics of Hawking radiation. Fourth—and more crucially—the fact that a radially infalling geodesic detector clicks at the horizon, and even records a finite temperature there, does not mean by itself that it perceives “Hawking radiation”. The local spacetime geometry is not stationary along a radial geodesic, and actually changes more and more rapidly as the horizon (and then the singularity) is approached; it could be that the temperature measured at the horizon is due to this (trivial) curvature effect, and *not* to the peculiar structure of the vacuum which is the true origin of Hawking radiation. Rather than “does a near-horizon detector record a non-zero temperature”, the question which should be addressed is therefore “does a near-horizon geodesic detector record a non-zero temperature that cannot be explained away by a time-varying gravitational potential”?

The purpose of this paper is to shed more light on these questions by reconsidering the response of geodesic UDW detectors during and after gravitational collapse. We address the four aforementioned concerns as follows:

1. We define and study a suitably defined time-dependent “temperature” function, as in [10], but

also related it explicitly to UDW response functions by an *adiabatic expansion*.

2. We only use globally defined coordinates, regular on the horizon as well as everywhere else (except at the singularity).
3. We avoid working with the extended Schwarzschild spacetime, where the effect of gravitational collapse must be mimicked by a suitable choice of vacuum state (e.g. the “Unruh vacuum” of [3] or the “collapse vacuum” of [10]). Instead, we consider gravitational collapse geometries and the corresponding—uniquely defined—in-vacuum state.
4. In a final section, we also consider generalized (non-Schwarzschild) black hole geometries, where the degeneracy between the surface gravity and the curvature scale at the horizon is lifted. This allows us to disentangle curvature effects and Hawking radiation at the horizon.

Another assumption made in [10] which we relax in this paper is that *only outgoing modes couple to UDW detectors*. In fact, we will see that there exists a class of infalling trajectories for which the response of UDW detectors is actually *dominated* by incoming modes. This is a somewhat surprising of our analysis, which, to our knowledge, was not anticipated in the literature.

The plan of the paper is as follows. In sec. II, we introduce the model of gravitational collapse used in the paper (a Vaidya ingoing shell), and discuss our “quasi-temperature formalism” to study the response of UDW detectors on arbitrary trajectories in that spacetime. We then apply this formalism in sec. III to various geodesic trajectories: circular orbits, radially infalling trajectories, and inspiral orbits. Our findings are compared with a flux computation in sec. IV. In the next section, we address in more details the role played by the local spacetime curvature on the response of near-horizon geodesic detectors, by means of an artificial black hole model where the curvature vanishes near the horizon. Sec. VI contains a discussion of our results and our conclusion. Details on adiabatic expansions of UDW-like response functions are given in the Appendix.

II. THE HAWKING EFFECT ALONG GENERAL TRAJECTORIES

The Hawking effect is the perception of an outgoing thermal flux at temperature $T_H \equiv (8\pi M)^{-1}$ by asymptotic inertial observers at rest relative to a Schwarzschild black hole. In this section we introduce a formalism to study response of particles detectors moving along

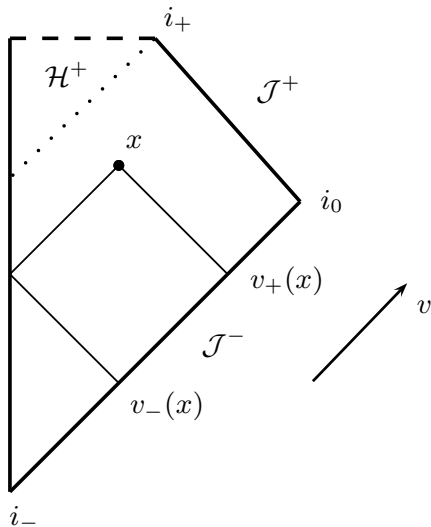


FIG. 1. Definition of the (globally defined) “eikonal coordinates” (v_+, v_-) for spherically-symmetric gravitational collapse.

more general trajectories. This allows us to consider two aspects of the Hawking phenomenon which are not often described in the literature: the *onset* of black hole evaporation after the horizon has formed, and the *time-dependence* of the spectra recorded by non-stationary detectors.

Throughout this paper, we shall make the following assumptions and approximations:

- We only consider minimally coupled massless scalar fields.
- We neglect the contribution of non-spherically symmetric ($l \neq 0$) field modes as well as all backscattering effects, so that the dynamics of the field is effectively two-dimensional.
- We neglect any backreaction of Hawking radiation on the background spacetime.

A. Collapse geometry and null coordinates

Our model gravitational collapse in this paper is the *Vaidya ingoing shell*. (Another collapse geometry will be discussed in sec. V.) As is well known, this spacetime consists of two patches separated by a null thin-shell: a flat region inside the shell, and a Schwarzschild region outside the shell. Its metric is conveniently written in Eddington-Finkelstein advanced coordinates (v, r) as

$$ds^2 = -\left(1 - \frac{r_s}{r}\Theta(v)\right)dv^2 + 2dvdr + r^2d\Omega^2, \quad (1)$$

where $r_s \equiv 2M$ is twice the mass of the shell (which is located at $v = 0$), $\Theta(v)$ is the Heaviside function and $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the standard angular metric. The surface gravity of the black hole is $\kappa \equiv (2r_s)^{-1}$.

To analyze the Hawking phenomenon in such a collapse model, it is useful to introduce globally defined *null* coordinates (v_+, v_-) for the (v, r) sector of spacetime. These are constructed as follows. For each event x , consider the two null rays (incoming and outgoing) meeting at x , and let $v_+(x)$ and $v_-(x)$ be their respective Eddington-Finkelstein advanced time at past null infinity, with $v_-(x) \leq v_+(x) = v(x)$ (see Penrose diagram in Fig. 1).

The physical interpretation of the (v_+, v_-) coordinates is tied to the eikonal approximation for the propagation of massless fields $\phi(x)$, according to which

$$\frac{\phi(x)}{r} \sim \lim_{r \rightarrow \infty} \frac{\phi(v_+(x), r) - \phi(v_-(x), r)}{r}. \quad (2)$$

In this approximation, the field at a given point x is expressed as the superposition of two spherical waves emanating from past null infinity \mathcal{J}^- : a convergent one, arriving directly from $v_+(v, r)$ (first term above), and a divergent one, arriving from $v_-(v, r)$ after a reflection off the origin (second term above), as in Fig. 1. (For a harmonic mode $e^{-i\omega v}$ on \mathcal{J}^- , the phases of these two waves at a given point x are $\omega v_{\pm}(x)$). For this reason, we call (v_+, v_-) the *eikonal coordinates* of x .

In the case of the Vaidya metric (1), it is possible to give the $(v, r) \mapsto (v_+, v_-)$ mapping in closed form. By construction, we have $v_+ = v$, and, when (v, r) is inside the shell, $v_-(v, r) = v - 2r$; for a point (v, r) outside the shell ($v > 0$), we must integrate the equation $ds^2 = 0$ from (v, r) back to the point $(\tilde{v} = 0, \tilde{r})$ where it meets the shell, and then write $v_-(v, r) = -2\tilde{r}$. This gives

$$v_-(v, r) = \begin{cases} v - 2r & \text{for } v < 0 \\ -2r_s \left[1 + W(\delta e^{\delta - \kappa v}) \right] & \text{for } v \geq 0, \end{cases} \quad (3)$$

where $\delta \equiv r/r_s - 1$ and $W(z)$ denotes the Lambert W -function, defined as the principal solution of $W(z)e^{W(z)} = z$. (For the reader's convenience, the graph of $W(z)$ plotted in Fig. 2.)

Since $W(z) \sim z$ as $z \rightarrow 0$, the identity (3) immediately gives the equation of the event horizon as $v_- = -2r_s$. But, more importantly, it also reveals the peculiar disruption of phase fronts induced by gravitational collapse: from (3) we see that v_- is constant where $\delta e^{\delta - \kappa v}$ is constant, and in particular $v_- \simeq -2r_s$ where $\delta e^{\delta - \kappa v} \ll 1$. Due to the peculiar analytic form of the function $\delta \mapsto \delta e^{\delta - \kappa v}$ for each $v \geq 0$, the region where $\delta e^{\delta - \kappa v} \ll 1$ is very sharply defined. Inside this region, which expands away from the horizon as v grows (until it eventually covers the whole spacetime), the phase of a vacuum fluctuation emerging from the shell after bouncing off its center

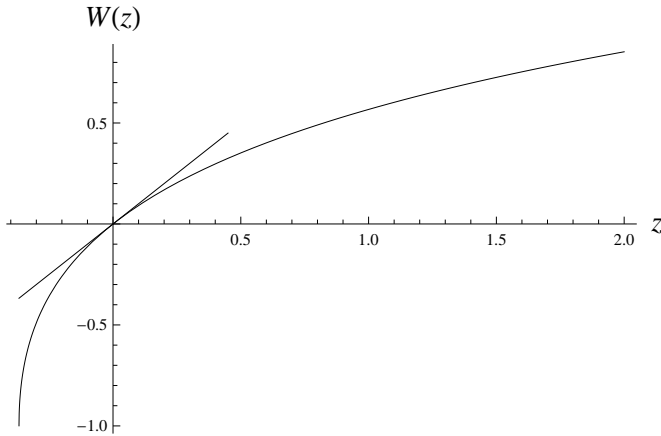


FIG. 2. The Lambert W -function, defined on $[e^{-1}, \infty)$ by $W(z)e^{W(z)} = z$ and such that $W(z) \sim z$ for $z \rightarrow 0$.

is *almost exactly equal* to $-2r_s$; this is where the Hawking phenomenon takes place. We shall call it the *Hawking region*. The level curves of $v_-(v, r)$ are represented in Fig. 3; in pictorial terms, Fig. 3 shows how gravitational collapse “opens up”, or perhaps “breaks open”, the in-vacuum.

Notice that, in the Hawking region, the eikonal coordinate v_- takes the same functional form as the Kruskal-Szekeres U coordinate for eternal black holes. Defining indeed, à la Kruskal-Szekeres, $U(v, r) \equiv -2r_s e^{-\kappa u(v, r)}$ with $u(v, r) \equiv v - 2r - 2r_s \log \delta$, from see from (3) and the asymptotic formula $W(z) \sim z$ as $z \rightarrow 0$ that

$$v_-(v, r) \sim U(v, r) - 2r_s \quad (4)$$

when $\delta e^{\delta - \kappa v} \ll 1$. Since eternal black holes are not real objects, the relation (4) can be thought of as providing the physical interpretation of Kruskal’s and Szekeres’ U coordinate: it captures the structure of outgoing phase fronts in the Hawking region of a (non-eternal) black hole.

B. Unruh-DeWitt detectors

Unruh-DeWitt (UDW) detectors [3, 11] are point-like monopole detectors, which measure the Wightman function $G(x, y)$ of the field along a given trajectory. Specifically, the response function of a UDW detector moving along the trajectory $\gamma(\tau)$ is, to first order in perturbation theory,

$$\mathcal{R}(\Omega) = 2 \operatorname{Re} \int_{-\infty}^{\infty} du \chi(u) \int_0^{\infty} ds \chi(u-s) e^{-i\Omega s} G(\gamma(u), \gamma(u-s)). \quad (5)$$

In this expression, Ω is the energy gap between the two stationary states of the detector, $G(\gamma(u), \gamma(u-s))$ is the pull-back of the Wightman function to the detector’s worldline, and $\chi(u)$ is a non-negative “switching” or “window” test function. The introduction of a switching function in the definition of the UDW response function ensures that $\mathcal{R}(\Omega)$ is well-defined and consistent with causality: choosing a switching function such that $\chi(u) \simeq 0$ for $u \geq \tau$ and $u \leq \tau - \Delta\tau$ ensures that $\mathcal{R}(\Omega)$ only depends on the time interval $\Delta\tau$ in the past of τ . Motivated by this causality issue, Svaiter and Svaiter introduced in [12] the simple ansatz $\chi(u) = \Theta(u - \tau - \Delta\tau)\Theta(\tau - u)$, which in the $\Delta\tau \rightarrow \infty$ limit leads to a transition rate (τ -derivative of the response function \mathcal{R})

$$\dot{\mathcal{R}}(\Omega) = 2 \operatorname{Re} \int_0^{\infty} ds e^{-i\Omega s} G(\gamma(\tau), \gamma(\tau-s)). \quad (6)$$

However appealing at first sight, this ansatz has two unpleasant features, which are due to the fact that $\chi(u) = \Theta(u - \tau - \Delta\tau)\Theta(\tau - u)$ is not a proper test function (it is not smooth): first, the standard $-i0$ prescription for the

Wightman function yields non-Lorentz-invariant transition rates [13] and must be replaced by a more complicated prescription [13–16]; second, the ultraviolet behavior (large Ω limit) is qualitatively different on stationary and non-stationary trajectory [17, 18]. Both these shortcomings disappear if a smooth switching function is used instead.

C. Wightman function in the (s -wave) in-vacuum

As mentioned earlier, in this paper we shall restrict our attention on the s -wave sector (spherically symmetric field configurations) of the in-vacuum. In this approximation, standard arguments [19, 20] show that $G(x, y)$ takes the logarithmic form

$$G(x, y) \propto \ln \left((v_+(x) - v_+(y) - i0)(v_-(x) - v_-(y) - i0) \right). \quad (7)$$

Using the relation (4), this expression can be identified in the Hawking region with the one given in [3] to describe

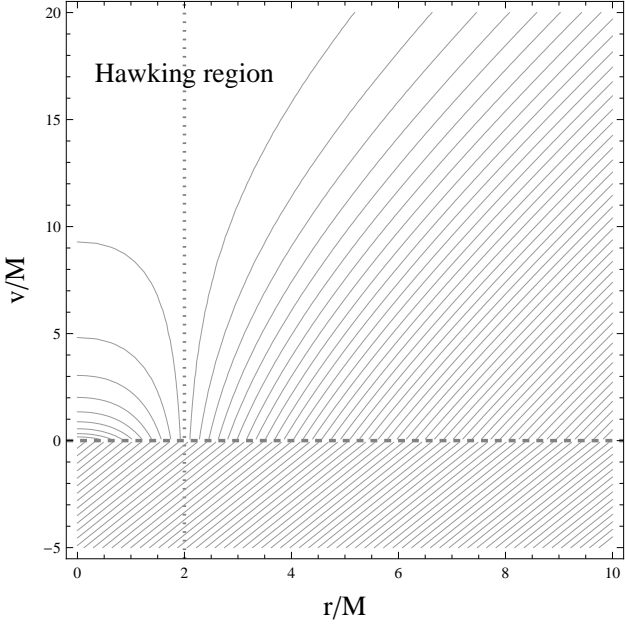


FIG. 3. “Portrait of the vacuum”: level curves of the eikonal coordinate $v_-(v, r)$ in the Vaidya spacetime. The dashed horizontal line is the collapsing shell, the dotted vertical line is the event horizon, and the region in the top-left corner is the “Hawking region”. Its boundary, defined by $v_-(v, r) = -2r - \epsilon$ for some ϵ , is very sharp, and may be thought of as a “wall”.

a radiating eternal black hole and known as the “Unruh vacuum”,

$$G_U(x, y) \propto \ln \left((v(x) - v(y) - i0)(U(x) - U(y) - i0) \right). \quad (8)$$

These states—the collapse vacuum (7) and the Unruh vacuum (8), in the s -wave sector—have a peculiar property, which is not shared by more general vacua: the incoming and outgoing modes couple to UDW detectors independently. This means that the UDW response function \mathcal{R} splits as $\mathcal{R} = \mathcal{R}_+ + \mathcal{R}_-$, where \mathcal{R}_\pm is given by (5) with $G(x, y)$ replaced by

$$\ln (v_\pm(\gamma(u)) - v_\pm(\gamma(u)) - i0). \quad (9)$$

The thermal properties of \mathcal{R}_+ and \mathcal{R}_- can thus be studied independently, and are completely determined by the behavior of each of eikonal coordinates v_\pm along the detector’s trajectory. In particular, in the case of stationary trajectories leading to a thermal spectrum, the thermality (aka detailed balance) condition for \mathcal{R}_+ and \mathcal{R}_- defines *two* temperatures T_+ and T_- , by

$$\mathcal{R}_\pm(-\Omega) = e^{\Omega/T_\pm} \mathcal{R}_\pm(\Omega). \quad (10)$$

These temperatures need not coincide in general. This is clear in the asymptotic Hawking effect, where $T_+ = 0$ and $T_- = T_H = \kappa/2\pi$.

D. Hawking temperature from the peeling of outgoing modes

The mechanism responsible for the perception of Hawking radiation has been described by many authors as an “exponential redshift” or “peeling” effect. After [2], this statement is usually expressed in terms of the canonical mapping from \mathcal{I}^+ to \mathcal{I}^- which relates the retarded and advanced Eddington-Finkelstein times of a given null ray. From the perspective of UDW detectors, however, it is more natural to state the “peeling” condition in a trajectory-dependent way, as follows: if $\gamma(\tau)$ is the trajectory of an asymptotic observer at rest relative to the hole, the quantity $v_-(\gamma(\tau))$ satisfies

$$-\frac{\ddot{v}(\gamma(\tau))}{\dot{v}(\gamma(\tau))} = \kappa. \quad (11)$$

(From now on, we shall drop the explicit reference to γ , the trajectory being understood from the context.). Integrating this condition gives

$$v_-(\tau) - v_-(\tau - s) \propto e^{-\kappa(\tau-s)/2} \sinh(\kappa s/2). \quad (12)$$

If we plug this relation into (9) and perform the standard residue integration [19], we arrive at the celebrated thermal spectrum at temperature $T_H = \kappa/2\pi$

$$\dot{\mathcal{R}}_-(\Omega) \propto \frac{1}{\Omega(e^{2\pi\Omega/\kappa} - 1)}. \quad (13)$$

(The $\Omega \rightarrow 0$ divergence is of course an artifact of the two-dimensional approximation.)

The main advantage of this trajectory-based formulation of the exponential redshift argument is that it applies *mutatis mutandis* to any trajectory where \ddot{v}_-/\dot{v}_- is a constant: for the same reasons that asymptotic observers such that $|\ddot{v}/\dot{v}| = \kappa$ perceive a thermal spectrum at temperature $\kappa/2\pi$, non-asymptotic observers such that $|\ddot{v}/\dot{v}|$ is constant perceive a thermal spectrum at temperature

$$T_- = \frac{1}{2\pi} \left| \frac{\ddot{v}_-}{\dot{v}_-} \right|. \quad (14)$$

This is the case e.g. for (late-time) static observers at any radius $r > r_s$: from the estimate (4), we compute

$$\left| \frac{\ddot{v}_-}{\dot{v}_-} \right| = \kappa \dot{v}(r). \quad (15)$$

The time-derivative $\dot{v}(r)$ along a static trajectory can be read off from the metric (1), as $\dot{v}(r) = (1 - r_s/r)^{-1/2}$, and therefore the temperature of outgoing modes measured by static UDW detector is readily found to be

$$T_-^{\text{stat}}(R) = T_H(1 - r_s/r)^{-1/2}. \quad (16)$$

Again, this is consistent with the Tolman equilibrium condition [5].

E. Non-static trajectories: quasi-temperature and adiabatic approximation

It has been argued in [10] that the *approximate* constancy of $T_-(\tau) \equiv |\ddot{v}_-(\tau)/\dot{v}_-(\tau)|/(2\pi)$, expressed as

$$\eta_- \equiv \left| \frac{\dot{T}_-}{T_-^2} \right| \ll 1, \quad (17)$$

is a sufficiently condition for the perception of thermal Hawking radiation. This “adiabaticity condition”, considered previously [21] in the context of evolving black holes, allows to study the response of UDW detectors along non-stationary trajectories [10] in a very straightforward way, simply by computing $T_-(\tau)$. Heuristically, when $\eta_- \ll 1$, the response function $\mathcal{R}(\tau, \Omega)$ of a UDW detector switched off (smoothly) at time τ will be indistinguishable from a thermal spectrum with τ -dependent temperature $T_-(\tau)$. In fact, even when η is not small, this relationship still holds in the ultraviolet limit, where $\Omega \gg T_-(\tau)$.

Following this rationale, we will call *quasi-temperature* of a time-dependent spectrum $\mathcal{R}(\tau, \Omega)$ a function $T(\tau)$ such that, at any given time τ , the detailed balance condition holds in the ultraviolet limit,

$$\mathcal{R}(\tau, -\Omega) \sim e^{\Omega/T(\tau)} \mathcal{R}(\tau, \Omega) \quad \text{for } |\Omega| \gg T_-(\tau). \quad (18)$$

See the Appendix for more details on the adiabatic approximation.

F. Two quasi-temperatures for Hawking radiation

Our approach to the study of Hawking radiation along general trajectories is based on this adiabatic approximation. For a given trajectory, we shall compute the quasi-temperatures of both outgoing and incoming modes as

$$T_{\pm}(\tau) = \frac{1}{2\pi} \left| \frac{\ddot{v}_{\pm}(\tau)}{\dot{v}_{\pm}(\tau)} \right|. \quad (19)$$

As we will see, these quasi-temperatures together with the corresponding adiabaticity parameters

$$\eta_- \equiv \left| \frac{\dot{T}_-}{T_-^2} \right| \ll 1, \quad (20)$$

provide a valuable handle on the response of UDW detectors to Hawking radiation, especially in the large Ω limit.

III. RESPONSE OF GEODESIC DETECTORS

In this section, we apply the quasi-temperature formalism to the case of geodesic trajectories in the

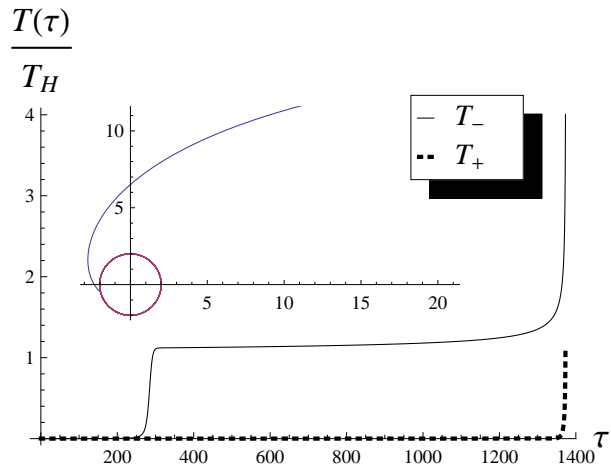


FIG. 4. The quasi-temperature of outgoing (continuous) and incoming (dashed) as a function of proper time along the inspiral geodesic $(E, L) = (1, 3)$ until it reaches the horizon. The inset shows the geodesic in the equatorial plane. The step at $\tau \simeq 300$ corresponds to the moment when the trajectory enters the Hawking region.

Schwarzschild region ($v > 0$) of the Vaidya collapse geometry described above.

A. Schwarzschild geodesics

Timelike Schwarzschild geodesics are characterized by two orbital parameters: their energy per unit rest mass E and their angular momentum per unit rest mass $L = r^2\dot{\varphi}$. In Eddington-Finkelstein coordinates, the geodesic equations read

$$\begin{cases} \dot{r}^2 + (1 - r_s/r)(1 + L^2/r^2) = E^2 \\ -(1 - r_s/r)\dot{v}^2 + 2\dot{v}\dot{r} + L^2/r^2 = -1. \end{cases} \quad (21)$$

From these equations and (3), we can in principle obtain $v_+ = v$, v_- and their derivatives—hence the quasi-temperatures T_{\pm} and the adiabaticity parameters η_{\pm} —along any geodesic (E, L) . This approach is illustrated in fig. 4 for one inspiral trajectory with $(E, L) = (1, 3)$, by means of a numerical integration of (21). This plot shows interesting phenomena: the setting off of Hawking radiation after gravitational collapse, the increase of the quasi-temperature of outgoing modes, *and* of that of incoming modes as the horizon is approached.

However, in order to get an *analytical* understanding of this plot, and in particular of the dependence of T_{\pm} and η_{\pm} on E and L , the best approach is to consider special regimes of interest (circular orbits, radial trajectories in the asymptotic and near-horizon regimes) separately.

This is the approach we shall take in the following. Our presentation will be rather pedestrian: we have found, indeed, that the details of the computation of T_{\pm} and η_{\pm} provide a good deal of insight into the inner workings of the Hawking phenomenon; they display in a very transparent way the interplay between the geometry of spacetime (and its geodesics) and the structure of the in-vacuum which is responsible for the evaporation of black holes.

B. Circular orbits

Circular orbits are characterized by their Schwarzschild radius $r > 3r_s/2$, or equivalently by $\delta = r/r_s - 1 > 1/2$.¹ By virtue of their stationarity, these orbits have constant \dot{v} , given by

$$\dot{v}(\delta) = \left(\frac{1 + \delta}{\delta - 1/2} \right)^{1/2}. \quad (22)$$

In particular, $\ddot{v} = 0$, hence $T_+ = 0$: incoming modes do not couple to UDW detectors on circular orbits. On the other, from (3) we get for the outgoing modes

$$\frac{\ddot{v}(\delta)}{\dot{v}(\delta)} = -\kappa \dot{v}(\delta) \left(1 + \delta e^{\delta - \kappa v} \frac{W''(\delta e^{\delta - \kappa v})}{W'(\delta e^{\delta - \kappa v})} \right). \quad (23)$$

Using (22) and the standard formula for the derivative of the Lambert W -function,

$$W'(z) = \frac{W(z)}{z(1 + W(z))}, \quad (24)$$

we arrive at

$$T_-(v, \delta) = T_H \left(\frac{1 + \delta}{\delta - 1/2} \right)^{1/2} \times \left| 1 - \frac{W(\delta e^{\delta - \kappa v})(2 + W(\delta e^{\delta - \kappa v}))}{(1 + W(\delta e^{\delta - \kappa v}))^2} \right|. \quad (25)$$

In this expression, the second term represents the transient regime before the orbit has been “swallowed” in the Hawking region. After this transient, $T_-(v, \delta)$ reaches the stationary value

$$T_-^{\text{circ}}(\delta) \equiv \lim_{v \rightarrow \infty} T_-(v, \delta) = T_H \left(\frac{1 + \delta}{\delta - 1/2} \right)^{1/2}. \quad (26)$$

¹ The corresponding energies and angular momenta are given by

$$E_{\text{circ}}(r) = r_s \left[\frac{\delta^2}{(\delta + 1)(\delta - 1/2)} \right]^{1/2}, \quad L_{\text{circ}}(r) = r_s \left[\frac{(1 + \delta)^2}{2(\delta - 1/2)} \right]^{1/2}.$$

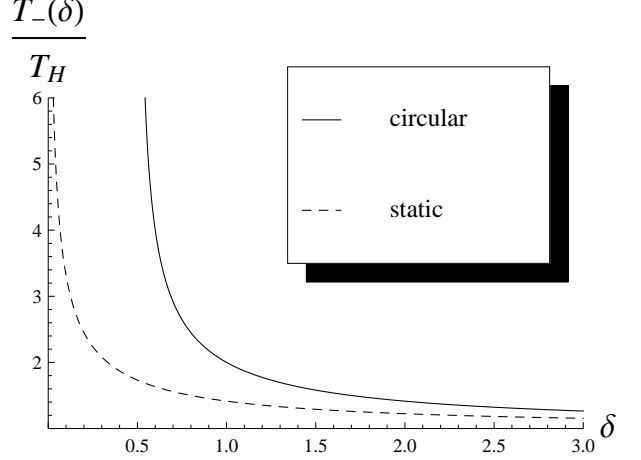


FIG. 5. Stationary trajectories: the temperature measured by static detectors (dashed line) and by geodesic detectors on a circular orbit (continuous line) around a Schwarzschild black hole as a function of $\delta = r/r_s - 1$.

Observe that, in spite of the fact that circular orbits have zero acceleration, this temperature is always *larger* than the temperature measured by static detectors at the same distance from the hole (compare with (16)), and diverges on the photon sphere ($r = 3r_s/2$). This can be interpreted as a transverse Doppler effect: as the photon sphere is approached, the relative velocity between circular and static trajectories tends to the speed of light. These results are illustrated in fig. 5.

C. Radial trajectories: surprises on the horizon

Radially infalling trajectories ($L = 0$, $\dot{r} < 0$, $\dot{v} > 0$) are parametrized by their energy E , with $E \geq 1$ corresponding to unbound states and $E < 1$ to bound states. (The limiting case $E = 1$ describes a detector dropped from infinity with zero velocity into the black hole.) In addition to the time-dependence of the Hawking phenomenon itself (the transient represented by v -dependent terms in (25)), these trajectories are intrinsically non-stationary: as the black hole is approached ($\delta \rightarrow 0$), the local Riemann curvature becomes larger and larger. To avoid mixing these two effects, from now on we shall assume that the trajectory is well inside the Hawking region, and focus on the time-dependent effects arising from the trajectory itself.

Outgoing modes. Consider first the outgoing modes. Using (3), we have

$$\begin{aligned} \frac{\ddot{v}_-}{\dot{v}_-} &\simeq \frac{(\delta e^{\delta-\kappa v})}{(\delta e^{\delta-\kappa v})} \\ &= \frac{\ddot{\delta}(1+\delta) + (\dot{\delta} - \kappa\dot{v})(2\dot{\delta} + \delta(\dot{\delta} - \kappa\dot{v})) - \kappa\delta\ddot{v}}{\dot{\delta} + \delta(\dot{\delta} - \kappa\dot{v})}. \end{aligned} \quad (27)$$

Let us focus on the aforementioned limiting cases: the asymptotic limit ($\delta \gg 1$) and at horizon-crossing ($\delta = 0$).

In the asymptotic limit (which requires $E \geq 1$ to exist), the system (21) gives $\dot{\delta} \sim -(E^2 - 1)^{1/2}/r_s$, $\dot{v} \sim E - (E^2 - 1)^{1/2}$, $\ddot{\delta} \simeq 0$, and $\ddot{v} \simeq 0$, hence from (27) with $\delta \gg 1$,

$$\left. \frac{\ddot{v}_-}{\dot{v}_-} \right|_{\text{asympt}} \simeq \dot{\delta} - \kappa\dot{v} = -\kappa(E + \sqrt{E^2 - 1}). \quad (28)$$

Thus, in this limit we get

$$T_-^{\text{asympt}}(E) = T_H(E + \sqrt{E^2 - 1}). \quad (29)$$

This formula is consistent with Hawking's original prediction, with a Doppler factor accounting for the relative motion between the detector and the black hole when $E > 1$. In this regime the adiabaticity parameter $\eta_-^{\text{asympt}}(E)$ is of course vanishingly small.

At horizon-crossing ($\delta = 0$), on the other hand, the equations of motion (21) give $\dot{\delta} = -E/r_s$ and $\dot{v} = 1/2E$, and, by differentiation of the radial equation, $\ddot{\delta} = -1/2r_s^2$. Plugging this into (27) now gives

$$\left. \frac{\ddot{v}_-}{\dot{v}_-} \right|_{\text{hor}} = \frac{\ddot{\delta} + 2\dot{\delta}(\dot{\delta} - \kappa\dot{v})}{\dot{\delta}} = -\frac{2E}{r_s} = -4\kappa E \quad (30)$$

and therefore

$$T_-^{\text{hor}}(E) = 4ET_H. \quad (31)$$

Thus, not only is the quasi-temperature not zero on the horizon, but for unbound states ($E \geq 1$) it is actually *larger* than the Hawking temperature perceived by static observers at infinity. The formula (31) is consistent with the result of [10].²

We have stressed that the near-horizon regime is not stationary, hence that whether (31) can be interpreted as a temperature depends on the value of $\eta_-(E)$ there. Further differentiation of (21) and (27) gives

$$\eta_-^{\text{hor}}(E) = \frac{\pi}{4} \left(2 + \frac{1}{E^2} \right). \quad (32)$$

This number is of order 1 for all unbound trajectories ($E \geq 1$), and never smaller than $\pi/2 \simeq 1.6$: detectors with frequency $\Omega \simeq T_-^{\text{hor}}(E)$ will *not* confuse the vacuum state with a thermal state. Large-frequency detectors ($\Omega \gg T_-^{\text{hor}}(E)$), on the other hand, will not be able to make this difference.

Incoming modes. Let us now consider the quasi-temperature of *incoming* modes. As we saw earlier, these do not couple to UDW detectors on stationary trajectories (static or circular); it may seem appropriate to assume that the same holds along radial trajectories. This is indeed the case in the asymptotic limit, where we have seen that $\ddot{v} \simeq 0$ —but what about the near-horizon regime?

We have obtained $\dot{v} = 1/2E$ on the horizon. To get the corresponding value of \ddot{v} , we differentiate the second equation of (21). On the horizon, this gives $\ddot{v} = -\kappa/4E^2$, and therefore

$$T_+^{\text{hor}}(E) = \frac{T_H}{2E}. \quad (33)$$

For unbound states ($E \geq 1$), this quasi-temperature is always smaller than that of outgoing modes. The situation is different for bound states ($E < 1$): for any $E < 1/\sqrt{8}$ we have $T_+^{\text{hor}}(E) > T_-^{\text{hor}}(E)$, i.e. the incoming modes *dominate* the outgoing modes, and in the limit where $E \rightarrow 0$ (detectors dropped with zero velocity from close to the horizon), the quasi-temperature of incoming modes *diverges*. Taking these modes into account, we can therefore conclude that *the slower it is approached, the “hotter” a black hole appears*. In our view, this surprising result is the main contribution of this work. We will comment further on this finding in sec. VI.

Again, the proper interpretation of this result should be derived from the value of the adiabaticity parameter. In this case, we get

$$\eta_+^{\text{hor}}(E) = 2\pi|1 - 8E^2|. \quad (34)$$

Thus, just like for outgoing modes, η_+^{hor} is bounded from below by a number of order one (actually $2\pi \simeq 6.2$): that is, the adiabatic approximation is *not* good in this regime, and (33) can only be interpreted as the ultraviolet decay rate of detector spectra.

D. Inspiral orbits: effect of angular momentum at horizon-crossing

It is not difficult to generalize the above results to inspiral trajectories with non-zero angular momentum. As before, the equations of motion (21) gives $\dot{\delta}$, $\ddot{\delta}$, etc., as functions of E and L in the various regimes, and using (27) it is a simpler matter to derive $T_{\pm}^{\text{hor}}(E, L)$. We do

² Considering an “observer freely falling from infinity”, they find that “in the last stages of his approach to the horizon, and surprisingly at first sight, the effective temperature rises reaching exactly four times Hawking’s temperature” [10].

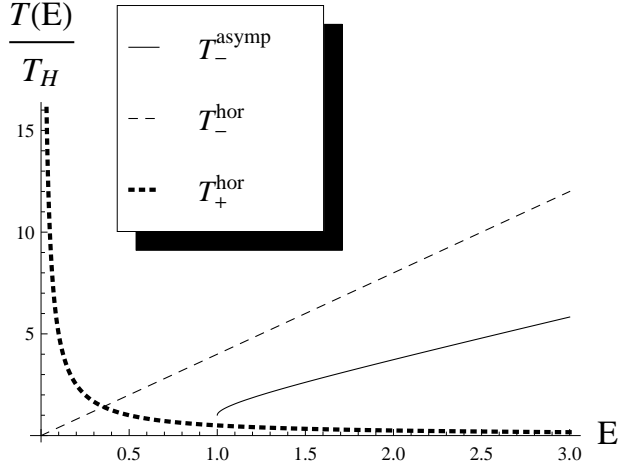


FIG. 6. Radially infalling trajectories: quasi-temperature of outgoing and incoming modes in different regimes, as functions of the energy E . For small energies $E \ll 1$ (bound states), incoming modes dominate.

not repeat this computation here; the results are

$$T_-^{\text{hor}}(E, L) = 4ET_H \quad (35)$$

and

$$T_+^{\text{hor}}(E, L) = \frac{T_H}{2E} \left| 1 + \frac{L^2}{r_s^2} \left(1 - \frac{8E^2}{1 + L^2/r_s^2} \right) \right| \quad (36)$$

and similar formulas for η_{\pm}^{hor} which we do not give here. Thus, while the quasi-temperature of outgoing modes at horizon crossing is independent of the angular momentum of the trajectory, the quasi-temperature of incoming modes is not. Instead, it is given by the above non-monotonous function of E and L .

IV. HAWKING FLUX ALONG RADIAL TRAJECTORIES

It is well-known that the response of UDW detectors to vacuum radiation is *a priori* independent from the existence of a non-zero energy-momentum tensor $\langle T_{ab} \rangle$. In the Hawking effect (unlike the Unruh effect), however, it turns out that $\langle T_{ab} \rangle \neq 0$ after the collapse: at future infinity, one finds $\langle T_{uu} \rangle = \pi T_H^2/12$ (and all other components zero), confirming that a Schwarzschild black hole does produce a stationary outgoing flux—that it actually *evaporates*.

What flux would infalling observers measure at horizon-crossing? Given the (surprising) results we have found for the response function of UDW detectors in the

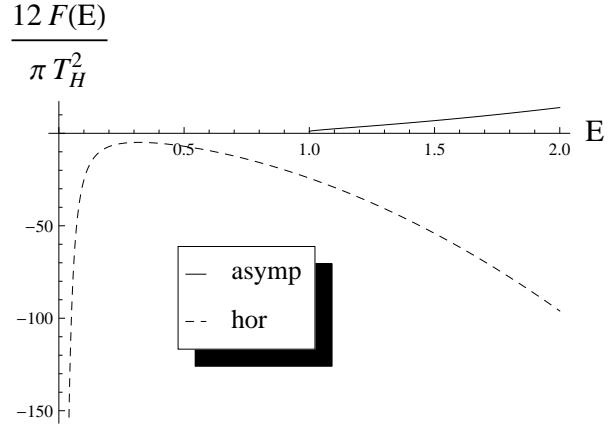


FIG. 7. The flux $\mathcal{F}(E)$ perceived by radially infalling observers with energy E , both in the asymptotic region (“asym”) and at horizon-crossing (“hor”). Compare with Fig. 6.

limit $E \rightarrow 0$ (also because they are based on an adiabatic expansion, which might seem unconvincing to some readers), it is interesting to consider the flux

$$\mathcal{F}(E) = \langle T_{ab} \rangle u^a n^b \quad (37)$$

along radially infalling trajectories $u^a(E)$ as a function of their energy E , say in the direction n^b orthogonal to u^a in the outwards direction. This quantity has been considered previously by several authors, see [22–24]. Is this quantity also divergent as $E \rightarrow 0$?

To answer this question, we can rely on standard results on vacuum energy-momentum tensor in two-dimensional spacetimes: given null coordinates (v_+, v_-) , the vacuum energy-momentum tensor can be written as [19, 25, 26]

$$\begin{aligned} \langle T_{v_{\pm} v_{\pm}} \rangle &= -\frac{1}{12\pi} C^{1/2} \partial_{v_{\pm}}^2 C^{-1/2} \\ \langle T_{v_+ v_-} \rangle &= \langle T_{v_- v_+} \rangle = \frac{RC}{96\pi}, \end{aligned} \quad (38)$$

where C is the conformal factor such that

$$ds^2 = -C(v_+, v_-) dv_+ dv_- \quad (39)$$

and $R = -4C^{-1} \partial_{v_+} \partial_{v_-} \ln C$ the two-dimensional scalar curvature.

In our case—which, by virtue of our assumption of spherical symmetry, is effectively two-dimensional—the factor $C(v_+, v_-)$ can be obtained from the Vaidya metric (1) by inverting the relation (3) between v_- and (v, r) . This gives

$$C(v_+, v_-) = \frac{a-1}{a} \frac{W(-ae^{-a+\kappa v_+})}{1+W(-ae^{-a+\kappa v_+})} \quad (40)$$

where $a = 1 + \kappa v_-$. Applying the formulas (38) and considering the $a \rightarrow 0$ limit for $\kappa v \gg 1$ (at late times), we compute

$$\begin{aligned}\langle T_{v_+ v_+} \rangle^{\text{hor}} &= -\frac{\pi T_H^2}{12} \\ \langle T_{v_- v_-} \rangle^{\text{hor}} &\sim \frac{\pi T_H^2}{2} e^{2\kappa v}.\end{aligned}\quad (41)$$

Note that the incoming component $\langle T_{v_+ v_+} \rangle$ at the horizon is minus the outgoing component $\langle T_{v_- v_-} \rangle$ at infinity.

Contracting (41) with the radial 4-velocity u^a (with components $\dot{v} = 1/2E$ and $\dot{v}_- = 2Ee^{-\kappa v}$ at the horizon) and its outwards-pointing unit normal n^b (with components $n_+ = 1/2E$ and $n_- = -2Ee^{-\kappa v}$), we arrive at

$$\mathcal{F}^{\text{hor}}(E) = -\pi T_H^2 \left(2E^2 + \frac{1}{48E^2} \right). \quad (42)$$

This describes an *ingoing* flux, which—like the quasi-temperature T_-^{hor} —diverges when $E \rightarrow 0$. This expression is to be compared with the flux at infinity, given by

$$\mathcal{F}^{\text{asympt}}(E) = \frac{\pi T_H^2}{12} \left(E + \sqrt{E^2 - 1} \right)^2. \quad (43)$$

We plot both functions of E in Fig. 7. Numerically, an observer dropped with zero velocity from e.g. $r = 100r_s/99$ (i.e. such that $E = 0.1$), the flux given by (42) is $\simeq 25$ times larger than the Hawking value $\pi T_H^2/12$.

V. ROLE OF CURVATURE IN NEAR-HORIZON HAWKING RADIATION

We have found that UDW detectors crossing the horizon of a Schwarzschild black hole will in general record non-zero quasi-temperatures as well as an ingoing flux. How much of this effect is due to the structure of the collapse vacuum, and how much to the mere fact that spacetime is curved on the horizon? It is impossible to answer this question within the framework of Schwarzschild black holes, because both their surface gravity (hence $T_H = \kappa/2\pi$) and their Riemann curvature on the horizon are controlled by the same parameter, namely their mass. To get some insight into the role played by curvature in the Hawking effect, it is therefore necessary to consider artificial black holes with non-Schwarzschild geometry, in the spirit of the “analogue gravity” program [27].

A. Artificial black holes with flat horizons

Since we are interested in models of gravitational collapse, we shall consider generalizations of the Vaidya met-

ric of the form

$$ds_f^2 = -\left(\Theta(-v) + \Theta(v)f(r) \right) dv^2 + 2dvdr + r^2 d\Omega^2 \quad (44)$$

where $f(r)$ could in principle be essentially any function of r . In particular, a function $f(r)$ such that (i) $f(r_s) = 0$, (ii) $f'(r_s) = 2\kappa$, (iii) $\lim_{r \rightarrow \infty} f(r) = 1$ and (iv) $\lim_{r \rightarrow 0} f(r) = -\infty$ will yield a black with the same asymptotic structure and surface gravity κ as the Schwarzschild black hole, but with a different Riemann curvature distribution. Here we will focus on an artificial collapse metric of the form $f(r) = (r/r_s - 1)\Psi(r)$, where $\Psi(r) \simeq 1$ near the horizon but ensures that conditions (iii, iv) above are satisfied. This corresponds to a spacetime where curvature (in the (v, r) sector) is concentrated on two locations: at the shell ($v = 0$), where it has a δ -function singularity, and at large radii (say $r \geq r_c$, where r_c is the extension of the flat region outside the horizon), where the derivatives of $\Psi(r)$ start taking non-zero values. So long as we only consider trajectories contained within the domain $\mathcal{F} \equiv \{v > 0, r < r_c\}$, we can think of this model as describing a “flat black hole”. Does this spacetime radiate in the same way as the Schwarzschild black hole?

Repeating the computation leading to (3) in the Vaidya case, we find for this “flat black hole”

$$v_-(v, r) = -2r_s(1 + \delta e^{-\kappa v}). \quad (45)$$

Just like the Vaidya case, this equation defines a “Hawking region” where $v_- \simeq -\kappa^{-1}$; the only difference difference is this region grows relatively faster than in the Vaidya case, as can be confirmed by comparing the conditions $\delta e^{-\kappa v} \ll 1$ (flat) and $\delta e^{\delta - \kappa v} \ll 1$ (Vaidya).

B. Quasi-temperatures at the flat horizon

To estimate the quasi-temperatures measured by UDW detectors in geodesic motion around this flat black hole, we repeat the steps in sec. III: write the geodesic equation for trajectories with orbital parameters (E, L) , which is now

$$\begin{cases} \dot{r}^2 + f(r)(1 + L^2/r^2) = E^2 \\ -f(r)\dot{v}^2 + 2\dot{v}\dot{r} + L^2/r^2 = -1, \end{cases} \quad (46)$$

and from (45) compute

$$\frac{\ddot{v}_-}{\dot{v}_-} = \frac{\ddot{\delta} - 2\kappa\dot{v}\dot{\delta} - \kappa\delta(\ddot{v} + \kappa\dot{v}^2)}{\dot{\delta} - \kappa\delta\dot{v}}. \quad (47)$$

Clearly, in the asymptotic limit, provided $f(r) \rightarrow 1$ for $r \rightarrow \infty$, these equations lead to the same results as in the Schwarzschild case. This confirms that asymptotic

Hawking radiation is a global phenomenon which does not depend on the actual curvature distribution around the black hole.

Consider however the case of UDW detectors in the near-horizon region, where $f(r) \simeq r/r_s - 1 = \delta$. In this regime, (46) gives the same values for $\dot{\delta}$, \dot{v} and $\ddot{\delta}$ as in the Schwarzschild case, and in particular $\ddot{\delta} = 2\kappa\dot{v}\dot{\delta}$, but now $\ddot{v} = -\kappa\dot{v}^2$. Thus, all terms on the numerator of (47) cancel, and therefore

$$T_-^{\text{near-hor}}(E, L) = 0. \quad (48)$$

The superscript indicates that this identity holds not just on the horizon, but in the whole near-horizon region where the black hole is flat. In this regime, the outgoing modes do *not* couple to UDW detectors. This shows that, for these to become thermal, there must be a region where the Riemann curvature is non-zero between the horizon and the observer (even if its actual distribution in space is irrelevant).

As for the incoming modes, the relationship $\ddot{v} = -\kappa\dot{v}^2$ gives

$$T_+^{\text{hor}}(E, L) = \frac{T_H}{2E} \frac{(L^2 + r_s^2)}{r_s^2}. \quad (49)$$

The incoming modes, which unlike the outgoing modes must have encountered some curvature on their way to the horizon, do couple to UDW detectors, even more so that L is large and E small.

We have seen at the end of the previous section that there exists an “irreducible non-adiabaticity” of horizon-crossing radial geodesics in the Schwarzschild spacetime. This “irreducible non-adiabaticity” is also present in the flat case, as can be confirmed by the computation of η_+^{hor} : sticking for simplicity to $L = 0$, we get

$$\eta_+^{\text{hor}}(E) = 2\pi. \quad (50)$$

This irreducible non-adiabaticity is thus independent of the local geometry of the horizon; it is an intrinsic feature of Hawking radiation as perceived by freely-falling near-horizon observers.

C. Flux at the horizon

We close this section by repeating the computation of the flux measured by infalling observers $\mathcal{F}(E) = \langle T_{ab} \rangle u^a n^b$. In the flat case, the relation (45) gives $C(v_+, v_-) = e^{\kappa v}$, hence from the Davies-Fulling-Unruh formula (38)

$$\langle T_{v_+ v_+} \rangle = -\frac{\pi T_H^2}{12}; \quad \langle T_{v_- v_-} \rangle = \langle T_{v_- v_+} \rangle = 0 \quad (51)$$

and therefore

$$\mathcal{F}^{\text{hor}}(E) = -\frac{\pi T_H^2}{48E^2}. \quad (52)$$

Again, we find that flattening the horizon cancels the effect of outgoing modes at the horizon, but not that of ingoing modes.

VI. DISCUSSION AND CONCLUSION

In this paper, we have considered Hawking radiation from the perspective of UDW detectors evolving on non-asymptotic trajectories. When the trajectory is not stationary, or just after the collapse, their response is not exactly thermal, but by using a “quasi-temperature” formalism based on a suitable adiabatic expansion, we have shown how to get a basic understanding of their response, especially in the large frequency limit. This quasi-temperature formalism is especially convenient from the computational perspective: formula (II E) is explicit and can be straightforwardly applied to any trajectory of interest.

Following this approach, we have obtained several interesting results, which disprove the notion that Hawking radiation reduces to Unruh radiation at the horizon, or that “the equivalence principle is restored” [4] there (that is, assuming that the Hawking effect violates it in some sense,³). They are:

- In spite of the fact that these trajectories are not accelerated, the temperature perceived on circular orbits is always higher than that on static trajectories at the same distance from the hole—it actually diverges on the photon sphere.
- In the near-horizon region, a freely-falling detector couples to both the outgoing *and* the incoming modes of the field, to which it associates two different quasi-temperatures,⁴ depending on its energy and angular momentum. Both can be arbitrarily high (though in different regimes: for fast and slow moving detectors respectively) and neither has a Dopplerian velocity-dependence relative to the hole. This singular behavior of Hawking radiation in the $E \rightarrow 0$ limit is confirmed by a flux computation.
- While the thermality of outgoing modes relative to near-horizon geodesic observers can be traced

³ We probably would not have described the Hawking effect in those terms anyhow.

⁴ At least in the s -wave approximation discussion in this paper.

back to the spacetime curvature, this is not the case for incoming modes: these appear thermal (both in terms of UDW response functions and of fluxes) to infallers also in spacetimes with flat horizons. In both cases, there exists an irreducible non-adiabaticity at the horizon, in the sense that $|\dot{T}_\pm| \sim T_\pm^2$ at the horizon.

In a nutshell: far from a no-particle vacuum, a detector dropped with zero velocity from near the horizon will perceive intense Hawking radiation coming from infinity—that is, from the sky.

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Appendix A: Adiabatic expansion of UDW-like response functions

The validity of the adiabatic approximation mentioned in sec. II E has been studied in some detail [17, 28, 29] in the context of non-uniformly accelerated UDW detectors in flat spacetime (approximate Unruh effect). In this setup, Barbado and Visser [29] have devised an “adiabatic expansion” for the UDW transition rates in terms of the quantities $a^{(n)}(\tau)/a(\tau)^{n+1}$, where $a(\tau)$ is the instantaneous acceleration of the detector.

In this appendix, we describe the adiabatic expansion of any functional of a slowly varying function $k(\tau)$ of the form

$$R(\tau, \Omega; k) \equiv \int_{-\infty}^{\infty} ds K(\Omega s) g(\Delta_{\tau,s}[k]). \quad (\text{A1})$$

where K, g are any functions such that the above integral is well-defined, and $\Delta_{\tau,s}[k]$ is the functional

$$\Delta_{\tau,s}[k] \equiv \frac{v(\tau) - v(\tau - s)}{\dot{v}(\tau)s}, \quad (\text{A2})$$

with $v(\tau)$ any solution of the differential equation

$$\ddot{v}(\tau) = -k(\tau)\dot{v}(\tau). \quad (\text{A3})$$

The UDW response functions of the kind discussed in this paper obviously fit this scheme, with $K(z) =$

$\exp(-iz)$, $g(z) = \ln(z)$, and k (divided by 2π) the quasi-temperature function.

The functional $R(\tau, \Omega; k)$ defined by (A1) may be seen as a function of τ, Ω , and the infinitely many derivatives $k^{(n)}(\tau)$ of k at a given instant τ , or—better—as Ω^{-1} times the dimensionless function

$$R(\nu_0, \nu_1, \dots) \equiv \int_{-\infty}^{\infty} du K(u) g(\Delta_{\tau,u/\Omega}[k]) \quad (\text{A4})$$

of the dimensionless ratios $\nu_n \equiv k^{(n)}(\tau)/\Omega^{n+1}$. If we denote $R_0(\nu_0)$ the value of $R(\nu_0, \nu_1, \dots)$ in the particular case where $k(\tau) = k$ is constant, the question we wish to answer is: in the general case where $k(\tau)$ is not constant, can $R(\nu_0, \nu_1, \dots)$ be systematically expanded about $R_0(\nu_0)$, with each corrective term depending on only finitely many ν_n 's?

The key idea to get started with this question, which we found in [29], consists in introducing a scaling parameter α and $k^{(\alpha)}(\tau - s) \equiv k(\tau - \alpha s)$. Denoting $\nu_n^{(\alpha)} \equiv \alpha^n \nu_n$ the corresponding scaled dimensionless ratios, we have

$$R(\nu_0, \nu_1, \dots) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m}{d\alpha^m} R(\nu_0^{(\alpha)}, \nu_1^{(\alpha)}, \dots) \Big|_{\alpha=0}. \quad (\text{A5})$$

Using Faà di Bruno's formula (chain rule for higher derivatives), we may write the m -th term in this expansion as

$$\frac{1}{m!} \sum_{j=0}^m \int_{-\infty}^{\infty} du K(u) g^{(j)}(\Delta_0) B_{m,j}(\Delta_1, \dots, \Delta_{m-j+1}). \quad (\text{A6})$$

where $B_{n,j}$ are the Bell polynomials⁵, and

$$\Delta_j \equiv \partial_\alpha^j \Delta_{\tau,u/\Omega}[k^{(\alpha)}] \Big|_{\alpha=0}. \quad (\text{A7})$$

In the next paragraph, we will show that Δ_j is given by

$$\Delta_j = \frac{\nu_j}{\nu_0^{j+1}} \delta_j(\nu_0 u) \quad (\text{A8})$$

for some function δ_j . This result calls for several comments:

- The m -th term in the Taylor expansion (A5), given by (A6), is a function $R_m(\nu_0, \dots, \nu_m)$ of the first $m+1$ parameters ν_n only, viz.

$$R(\nu_0, \nu_1, \dots) = \sum_{m=0}^{\infty} R_m(\nu_0, \dots, \nu_m). \quad (\text{A9})$$

⁵ The Bell polynomials are defined by $B_{m,k}(z_1, \dots, z_{m-k+1}) = m! \sum_{i=1}^{m-k+1} \frac{1}{j_i!} (\frac{z_i}{i!})^{j_i}$, where the sum runs over all sequences of non-negative integers (j_1, \dots, j_{m-k+1}) such that $j_1 + \dots + j_{m-k+1} = k$ and $j_1 + 2j_2 + \dots + (m-k+1)j_{m-k+1} = m$.

- The relevant parameters controlling the convergence of the adiabatic expansion (A9) are indeed $\nu_j/\nu_0^{j+1} = k^{(j)}(\tau)/k(\tau)^{j+1}$, as observed in [29] in the context of the Unruh effect; in particular, we have for each $m \geq 1$

$$R_m(\nu_0, \dots, \nu_m) = \mathcal{O}\left(\sup_{1 \leq j \leq m} \frac{\nu_j}{\nu_0^{j+1}}\right). \quad (\text{A10})$$

When $k(\tau)$ is a polynomial function of τ (so that only finitely many ν_n 's are non-vanishing), we are assured that

$$R(\nu_0, \nu_1, \dots) = R_0(\nu_0) + \mathcal{O}\left(\sup_{j \geq 1} \frac{\nu_j}{\nu_0^{j+1}}\right). \quad (\text{A11})$$

In other words, the adiabatic approximation is good when all the expansion parameters ν_j/ν_0^{j+1} are small.

- The adiabatic expansion is not a high-frequency expansion; indeed, the functions $\delta_j(\nu_0 u) = \delta_j(k(\tau)u/\Omega)$ can well be non-polynomial in $1/\Omega$.

To prove (A8), we begin by expressing the argument of g in (A4) explicitly in terms of the dimensionless ratios ν_n 's. From (A3), we have

$$\Delta_{\tau, u/\Omega}[k] = \frac{\Omega}{u} \int_{-u/\Omega}^0 d\sigma \exp \int_{\sigma}^0 d\rho k(\tau + \rho). \quad (\text{A12})$$

Expanding $k(\tau + \rho)$ in powers of ρ and making the change of variable $\mu = \Omega\sigma$ in the σ -integral, this gives

$$\Delta_{\tau, u/\Omega}[k] = \frac{1}{u} \int_0^u d\mu \exp \sum_{n=0}^{\infty} \frac{\mu^{n+1} \nu_n}{(n+1)!}. \quad (\text{A13})$$

Next, we consider the effect of scaling by α on (A17),

$$\Delta_{\tau, u/\Omega}[k^{(\alpha)}] = \frac{1}{u} \int_0^u d\mu \exp \sum_{n=0}^{\infty} \frac{\mu^{n+1} \alpha^n \nu_n}{(n+1)!} \quad (\text{A14})$$

and compute its j -th derivative with respect to α , as in (A17). The 0-th derivative is easily found to be

$$\Delta_0 = \frac{e^{\nu_0 u} - 1}{\nu_0 u} \quad (\text{A15})$$

and for, $j \geq 1$,

$$\Delta_j = \frac{\nu_j}{j+1} \partial_{\nu_0}^{j+1} \Delta_0. \quad (\text{A16})$$

Computing explicitly the derivatives of (A15), we arrive at (A8) with

$$\delta_j(z) = \frac{(-1)^{j+1} j!}{z} \left(e^z \sum_{r=0}^{j+1} \frac{(-1)^r}{r!} z^r - 1 \right). \quad (\text{A17})$$

We close this appendix by evaluating explicitly the first corrective term R_1 in the particular case where $K(z) = e^{-iz}$ and $g(z) = \ln z$, namely

$$R(\tau, \Omega; k] = \Omega^{-1} R(\nu_0, \dots) \equiv \int_{-\infty}^{\infty} ds e^{-i\Omega s} \ln(\Delta_{\tau, s}[k]). \quad (\text{A18})$$

The real part of this integral is the UDW response function along a trajectory with quasi-temperature $T = k/2\pi$, studied in the main body of this paper. As is well-known, the 0-th order $R_0(\nu_0)$ is the standard two-dimensional thermal spectrum

$$R_0(\nu_0) = \frac{2\pi}{e^{2\pi/\nu_0} - 1}. \quad (\text{A19})$$

From (A6), (A7) and (A17), we have

$$R_1(\nu_0, \nu_1) = \frac{\nu_1}{\nu_0^2} \times \int_{-\infty}^{\infty} du e^{-iu} \frac{e^{\nu_0 u} (1 - \nu_0 u + (\nu_0 u)^2/2) - 1}{e^{\nu_0 u} - 1} \quad (\text{A20})$$

The integrand has simple poles at $u_n = 2i\pi n/\nu_0$ for each integer $n \neq 0$. Closing the contour in the lower half-plane and summing over $n \leq -1$, we obtain

$$R_1(\nu_0, \nu_1) = \frac{\nu_1}{\nu_0^2} \left(\frac{\pi^2 (1 + i\pi \coth(\pi/\nu_0))}{\nu_0 \sinh^2(\pi/\nu_0)} \right). \quad (\text{A21})$$

Reinstating the dimensionful variables $k(\tau)$, $\dot{k}(\tau)$ and Ω and writing $T = k/2\pi$, this gives

$$\text{Re}\{R(\tau, \Omega; k = 2\pi T]\} = \frac{2\pi}{\Omega(e^{\Omega/T} - 1)} \left(1 + \right. \quad (\text{A22})$$

$$\left. \frac{\dot{T}(\tau)}{T(\tau)^2} \frac{\Omega}{8\pi T(\tau)} \frac{e^{\Omega/2T(\tau)}}{\sinh^2(\Omega/2T(\tau))} + \dots \right). \quad (\text{A23})$$

Remarkably, the second term in the brackets is *exponentially* vanishing in the ultraviolet limit, showing that the adiabatic approximation is excellent in this regime, whatever the value of the parameter $\dot{T}(\tau)/T(\tau)^2$.

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